

Trigonometric Identities

Basic Identities:

$$\sin = \frac{y}{r} \quad (1)$$

$$\cos = \frac{x}{r} \quad (2)$$

$$\tan = \frac{y}{x} = \frac{\sin}{\cos} \quad (3)$$

$$\cot = \frac{x}{y} = \frac{\cos}{\sin} = \frac{1}{\tan} \quad (4)$$

$$\sec = \frac{r}{x} = \frac{1}{\cos} \quad (5)$$

$$\csc = \frac{r}{y} = \frac{1}{\sin} \quad (6)$$

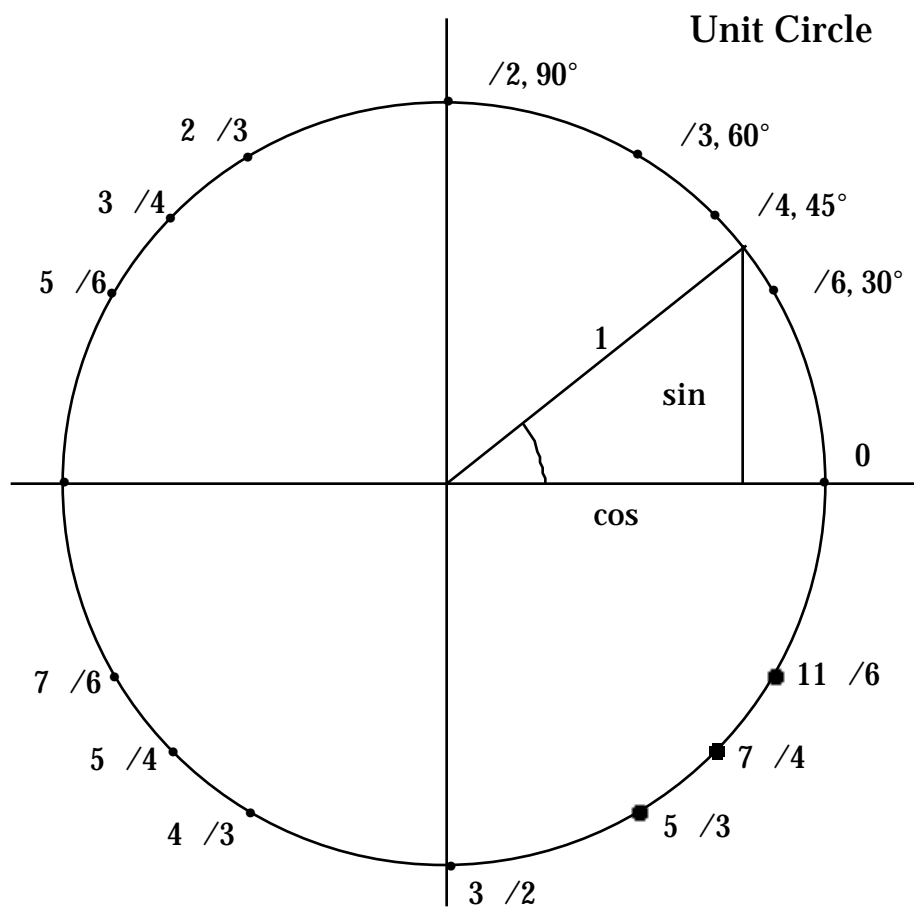
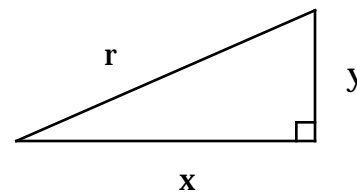


Table of Values from 0 to 2 :

degrees	0	30	45	60	90	120	135	150	180
radians	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	$2\pi/3$	$3\pi/4$	$5\pi/6$	
sin (y/r)	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
cos (x/r)	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	-1
tan (y/x)	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	\pm	$-\sqrt{3}$	-1	$\frac{\sqrt{3}}{3}$	0
cot (x/y)	\pm	$\sqrt{3}$	1	$\frac{\sqrt{3}}{3}$	0	$\frac{\sqrt{3}}{3}$	-1	$-\sqrt{3}$	\pm
sec (r/x)	1	$\frac{2\sqrt{3}}{3}$	$\sqrt{2}$	2	\pm	-2	$-\sqrt{2}$	$\frac{-2\sqrt{3}}{3}$	-1
csc (r/y)	\pm	2	$\sqrt{2}$	$\frac{2\sqrt{3}}{3}$	1	$\frac{2\sqrt{3}}{3}$	$\sqrt{2}$	2	\pm

Table continued

degrees	210	225	240	270	300	315	330	360
radians	$7\pi/6$	$5\pi/4$	$4\pi/3$	$3\pi/2$	$5\pi/3$	$7\pi/4$	$11\pi/6$	2
sin (y/r)	$\frac{1}{2}$	$\frac{-\sqrt{2}}{2}$	$\frac{-\sqrt{3}}{2}$	-1	$\frac{-\sqrt{3}}{2}$	$\frac{-\sqrt{2}}{2}$	$\frac{1}{2}$	0
cos (x/r)	$\frac{-\sqrt{3}}{2}$	$\frac{-\sqrt{2}}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
tan (y/x)	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	\pm	$-\sqrt{3}$	-1	$\frac{-\sqrt{3}}{3}$	0
cot (x/y)	$\sqrt{3}$	1	$\frac{\sqrt{3}}{3}$	0	$\frac{-\sqrt{3}}{3}$	-1	$-\sqrt{3}$	\pm
sec (r/x)	$\frac{-2\sqrt{3}}{3}$	$-\sqrt{2}$	-2	\pm	2	$\sqrt{2}$	$\frac{2\sqrt{3}}{3}$	1
csc (r/y)	-2	$-\sqrt{2}$	$\frac{-2\sqrt{3}}{3}$	-1	$\frac{-2\sqrt{3}}{3}$	$-\sqrt{2}$	-2	\pm

Some useful trigonometric relationships:

$$\sin^2 + \cos^2 = 1 \quad (7)$$

$$1 + \tan^2 = \sec^2 \quad (\text{divide [7] by } \sin^2) \quad (8)$$

$$1 + \cot^2 = \csc^2 \quad (\text{divide [7] by } \cos^2) \quad (9)$$

$$\sin(\pm) = \sin \cos \pm \cos \sin \quad (10)$$

$$\cos(\pm) = \cos \cos \mp \sin \sin \quad (11)$$

$$\tan(\pm) = \frac{\tan \pm \tan}{1 \mp \tan \tan} \quad (12)$$

$$\sin^2 = \frac{1}{2}(1 - \cos 2) \quad (13)$$

$$\cos^2 = \frac{1}{2}(1 + \cos 2) \quad (14)$$

$$\sin 2 = 2\sin \cos \quad (15)$$

$$\cos 2q = \cos^2 q - \sin^2 q = 2\cos^2 q - 1 = 1 - 2\sin^2 q \quad (16)$$

$$\tan 2 = \frac{2 \tan}{1 - \tan^2} \quad (17)$$

$$\sin \frac{1}{2} = \pm \sqrt{\frac{1 - \cos}{2}} \quad (18)$$

$$\cos \frac{1}{2} = \pm \sqrt{\frac{1 + \cos}{2}} \quad (19)$$

$$\tan \frac{\theta}{2} = \frac{1 - \cos \theta}{\sin \theta} = \frac{\sin \theta}{1 + \cos \theta} \quad (20)$$

All of the above relationships are easily proved from Euler's identity

$$e^{i\theta} = \cos \theta + i \sin \theta, \quad (21a)$$

and it also follows that

$$e^{-i\theta} = \cos \theta - i \sin \theta, \quad (21b)$$

$$\cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2} = \cos(-\theta) \quad (22)$$

$$\sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} = -\sin(-\theta) \quad (23)$$

and these identities can be manipulated to get a new and sometimes more convenient expression for the trigonometric function of an angle. Just in case you doubt this method, we append some derivations:

$$\begin{aligned} \cos^2 \theta + \sin^2 \theta &= \left(\frac{e^{i\theta} + e^{-i\theta}}{2} \right)^2 + \left(\frac{e^{i\theta} - e^{-i\theta}}{2i} \right)^2 \\ &= \frac{e^{2i\theta} + 2 + e^{-2i\theta}}{4} + \frac{e^{2i\theta} - 2 + e^{-2i\theta}}{-4} \\ &= \frac{e^{2i\theta} + 2 + e^{-2i\theta} - e^{2i\theta} + 2 - e^{-2i\theta}}{4} \\ &= 1. \end{aligned} \quad (24)$$

$$\sin 2\theta = \frac{e^{2i\theta} - e^{-2i\theta}}{2i}$$

$$\begin{aligned}
&= \frac{(e^i)^2 - (e^{-i})^2}{2i} \\
&= \frac{(e^i + e^{-i})(e^i - e^{-i})}{2i} \\
&= (e^i + e^{-i}) \sin \\
&= 2 \cos \sin .
\end{aligned} \tag{25}$$

The hyperbolic functions are analogous to the trig functions and often arise in physical situations. Their relations to the trig functions are as follows:

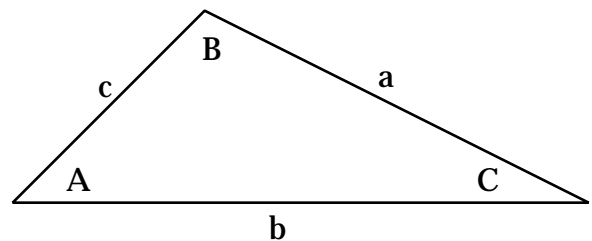
$$\sinh x = \frac{e^x - e^{-x}}{2} = \frac{e^{-i(ix)} - e^{i(ix)}}{2} = -i \sin(ix), \tag{26}$$

$$\cosh x = \frac{e^x + e^{-x}}{2} = \frac{e^{-i(ix)} + e^{i(ix)}}{2} = \cos(ix), \tag{27}$$

$$\cos x = \cosh \frac{x}{i} = \cosh(-ix) = \cosh(ix), \tag{28}$$

$$\sin x = i \sinh \frac{x}{i} = i \sinh(-ix) = -i \sinh(ix). \tag{29}$$

(The following laws apply for all triangles with angles, A, B and C and opposite side lengths as defined in the figure.)



$$\text{Law of Sines: } \frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} \tag{30}$$

$$\text{Law of Cosines: } c^2 = a^2 + b^2 - 2ab \cos C \tag{31}$$

Derivatives of Trig Functions:

$$\frac{d}{dx} (\sin u) = \cos u \left(\frac{du}{dx} \right) \quad (32)$$

$$\frac{d}{dx} (\cos u) = -\sin u \left(\frac{du}{dx} \right) \quad (33)$$

$$\frac{d}{dx} (\tan u) = \sec^2 u \left(\frac{du}{dx} \right) \quad (34)$$

$$\frac{d}{dx} (\cot u) = -\csc^2 u \left(\frac{du}{dx} \right) \quad (35)$$

$$\frac{d}{dx} (\sec u) = \sec u \tan u \left(\frac{du}{dx} \right) \quad (36)$$

$$\frac{d}{dx} (\csc u) = -\csc u \cot u \left(\frac{du}{dx} \right) \quad (37)$$

References: Shenk, Calculus and Analytic Geometry

Boas, Mathematical Methods in the Physical Sciences